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A Negative Theorem on Superpositions

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Let V be a bounded open set in Euclidean space R^n ($n \geqslant 2$) and $f_1, ..., f_r$ continuously differentiable mappings of V into R^{n-1}, of rank $n-1$ everywhere on V. (This is not essential, as explained in [4].) The set $S(f_1, ..., f_r)$ consisting of all sums $\sum g_\kappa \circ f_\kappa$, wherein each g_κ is bounded and continuous on all of R^{n-1}, is the linear space of *superpositions*. When E is a compact subset of V, we say that $S(f_1, ..., f_r)$ covers the Banach space $C(E)$, if the restriction of the functions in S, to E, represent all continuous real functions on E. This never occurs when E has interior points, and the same analysis is valid when E has positive Lebesgue measure [4]. A more interesting theorem concerns sets E with this negative property in relation to $S(f_1, ..., f_r)$, but favorably disposed for representation of continuous functions by superposition.

THEOREM. *There exists a compact set $E \subseteq V$, on which $S(f_1, ..., f_r)$ fails to cover C, while all mappings of class $C^1(V^-)$, except a set of first category, are one–one on E.*

In proving that $S(f_1, ..., f_r)$ is of first category in $C(E)$ we find a measure $\mu \geqslant 0$, $\mu \neq 0$, carried by E, and unimodular continuous function ψ_j, so that

$$\left| \int \psi_j(x)\, g \circ f_\kappa(x)\, \mu(dx) \right| \leqslant \epsilon_j \| g \|_\infty, \qquad 1 \leqslant \kappa \leqslant r,$$

with $\lim \epsilon_j = 0$. The first step in formation of a set E of this type, in combination with the positive property relative to $C^1(V^-)$, is selection of a special set of basis vectors in R^n. Let $\Gamma_1(x), ..., \Gamma_r(x)$ be the $(n-1)$-dimensional subspaces of R^n, spanned by the Jacobian matrices of $f_1, ..., f_r$ at x. Then each Γ_κ depends continuously on x, when construed as an element of a certain (Grassmannian) manifold of subspaces, whence the measurability of certain sets is assured, and the validity of Fubini's theorem.

LEMMA. *There exists a basis* $u_1, \ldots, u_n$ *of* R^n *and a set* N *of measure* 0 *in* V *such that for each integral combination* $p_1u_1 + \cdots + p_nu_n \neq 0$,

$$p_1u_1 + \cdots + p_nu_n \notin \cup \Gamma_\kappa(x), \qquad \textit{unless } x \in N.$$

Proof. The vectors $u_1, \ldots, u_n$ are chosen from the space $R^{n^2} = R^n \times \cdots \times R^n$ so that almost all choices $U = (u_1, \ldots, u_n)$ from a basis for R^n. Let Γ be a set of Lebesgue measure 0 in R^n, then the set

$$U: p_1x_1 + \cdots + p_nx_n \in \Gamma$$

has measure 0 in R^{n^2}, provided $|p_1| + \cdots + |p_n| > 0$. Setting $\Gamma = \bigcup \Gamma_\kappa(x)$, for x in V, we obtain the lemma by applying Fubini's theorem to the space $R^{n^2} \times V$.

Henceforth $u_1, \ldots, u_n$ is a special basis and H a compact subset of $V \sim N$. Let φ be an element of the Lebesgue space $L^1(H) \subseteq L^1(V)$ and $w = p_1u_1 + \cdots + p_nu_n \neq 0$. The mapping $F = (w \cdot x, f_\kappa(x))$ into $R^1 \times R^{n-1} = R^n$ is nonsingular everywhere on H. The implicit function theorem and a change of variables, lead to an identity

$$\int_{R^n} h(w \cdot x)\, g \circ f_\kappa(x) \cdot \varphi(x)\, dx = \int_{R^n} h(y_1)\, g(y_2, \ldots, y_n)\, \tilde{\varphi}(y)\, dy$$

for all bounded functions h on R^1 and g on R^{n-1}, where $\tilde{\varphi}$ depends on φ, w, κ and $\|\tilde{\varphi}\|_1 = \|\varphi\|_1$. From this we obtain the basic inequality.

$$\left|\int h(w \cdot x)\, g \circ f_\kappa \cdot \varphi(x)\, dx\right| \leqslant \int \left|\int h(y_1)\, \tilde{\varphi}(y_1, \ldots, y_r)\, dy_1\right| dy_2 \cdots dy_n$$

if $|g| \leqslant 1$ on R^{n-1}. If we choose $h_\lambda(t) = e^{i\lambda t}$, then the integral on the right tends to 0 as $\lambda \to +\infty$.

Let φ_0 be the characteristic function of H; for large enough λ we have

$$\left|\int h_\lambda(u_1 \cdot x)\, g \circ f_\kappa(x) \cdot \varphi_0(x)\, dx\right| < \|g\|_\infty/4, \qquad 1 \leqslant \kappa \leqslant r.$$

Our plan is to replace the measure $d\mu_0(x) = \varphi_0(x)\, dx$, by a sequence of measures μ_s, whose closed supports decrease to the required set E. After s steps we attain a function $\varphi_s > 0$ with these properties:

$$\|\varphi_0\|_1 < 2\, \|\varphi_s\|_1 < 4\, \|\varphi_0\|_1, \tag{1}$$

$$\left|\int h_{\lambda_j}(u_1 \cdot x)\, g \circ f_\kappa \cdot \varphi_s(x)\, dx\right| \leqslant \epsilon_j\, \|g\|_\infty, \qquad 1 \leqslant \kappa \leqslant r, \tag{2}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_s$ are real and $\epsilon_j < 2^{-j-1}$, $0 \leqslant j \leqslant s$. The functions h_{λ_j} are, of course, the ψ_j's mentioned before.

To continue the process, of constructing our sequence of measures and functions, we take a function $T \geqslant 0$, of period 2π in each variable and of class $C^\infty(R^n)$, and of mean-value 1, and (most important) $T(t_1, \ldots, t_n) = 0$ unless $| t_\kappa | < 2^{-s}$ (modulo 2π) for each coordinate. We shall set $\varphi_{s+1} = T(Yu_1 \cdot x, \ldots, Yu_n \cdot x)\, \varphi_s$ for large Y so that $\varphi_{s+1} \geqslant 0$ and the closed support of μ_{s+1} is contained in that of μ_s. Now T admits an absolutely convergent Fourier expansion

$$T(t) - 1 = \sum{}' a(p_1, \ldots, p_n) \exp i(p_1 t_1 + \cdots + p_n t_n).$$

Observe that if we replace t_κ by $Yu_\kappa \cdot x$ the sum can then be abbreviated $\sum_w' a(w) \exp iY(w \cdot x)$, wherein w takes all nonzero values $p_1 u_1 + \cdots + p_n u_n$. In combination with the absolute convergence $\sum | a(w)| < \infty$ and calculations made above, this leads to inequalities (for $1 \leqslant \kappa \leqslant r$, $0 \leqslant j \leqslant s$

$$\left| \int [T(Yu_1 \cdot x, \ldots, Yu_n \cdot x) - 1]\, g \circ f_\kappa(x) \cdot \varphi_s(x)\, \psi_j(x)\, dx \right|$$
$$< \eta(Y) \| g \|_\infty, \qquad \text{where } \eta(\infty) = 0.$$

For large numbers Y, this yields the first $s + 1$ inequalities of (2) for φ_{s+1} with $\epsilon_j' < 2^{-j-1}$, and we obtain the last inequality by fixing Y and choosing $\psi_{s+1} = \exp(\lambda u_1 \cdot x)$ for a large number λ_{s+1}. To attain the inequalities on $\| \varphi_{s+1} \|_1$ we have merely to write $g = 1$ and replace ψ_j by 1.

The weak* limit of the sequence (μ_s) obviously has the negative property required of μ, with the given sequence ψ_j. Moreover, on the support E of μ

$$| y_s u_1 \cdot x | \leqslant 2^{-s}, \ldots, | y_s u_n \cdot x | \leqslant 2^{-s} \qquad \text{(modulo } 2\pi)$$

for a sequence $y_s \to +\infty$. For $n = 2$, sets with this property were investigated in [2] under the name *Dirichlet sets.* It was proved that $C^1(V^-)$ contains a residual set of mappings, one–one on E; in fact for each f in $C^1(V^-)$ there is a g of this type, so that the first partial derivatives of f–g vanish everywhere on E. In particular, f can be one of the coordinates of f_1. The analysis for the case $n = 2$ applies with few changes in general, and the proof is thereby complete.

A somewhat different method allows us to construct a set E_1, so that $S(f_1, \ldots, f_r)$ does not cover $C(E_1)$, but some linear form L is one–one on E_1. (Hence L is real-analytic, while the category method of [2] produces only C^1 functions.) The proof of this consists in constructing a set E_1 with a far stronger property (borrowed from Fourier analysis [5, Chap. 5; 1, Chap. 7]): each continuous function on E_1, of modulus 1, is a uniform limit of exponentials $e(x) = e^{iyL(x)}$. To prove this we have only to find an open set on which each mapping $(L, f_1), \ldots, (L, f_r)$ is a diffeomorphism into R^n. The formation of E_1, and the proof that $S(f_1, \ldots, f_r)$ does not cover $C(E_1)$, then follows the inductive procedure of [3].

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